Optimal holdings of active, passive and smart beta strategies

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Abstract

The growing dominance of the core and explore model – a large passive index combined with a collection of high tracking error satellite portfolios – in conjunction with the growth of factor investing has renewed interest in how to allocate among different equity strategies. We study this problem from an expected shortfall perspective and find that portfolios that minimize expected shortfall differ substantially from portfolios generated using conventional methods.

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The past decade has witnessed a marked shift in the composition of institutional equity portfolios. Traditional style boxes and a heavy reliance on active management have made way for large allocations to a passive core combined with high tracking error, high conviction, satellite strategies – the core and explore approach. More recently, the emergence of low fee, factor based strategies has further upended this trend. These changes raise the natural question: how should an asset owner allocate among different equity strategies? This basic question goes beyond the active passive debate. After all, factor (e.g. smart beta) strategies have blurred the line between these two approaches.

Our main contribution is to answer this question using an allocation framework based on expected shortfall. We find that the portfolios generated by this framework differ substantially from the portfolios generated using conventional methods.

One way to understand the appeal of core and explore is to adopt a policy portfolio approach and to think in terms of expected excess return relative to the policy benchmark.[1] Benchmarking is pervasive in investment management (see Grinold and Kahn (2000), pages 88-89), in part because it facilitates the monitoring of the principal-agent relationship between asset owner and asset manager. As a result, in this paper we focus on relative performance, and assume that the only objective is to outperform the benchmark.

A typical domestic equity core and explore model has about two thirds in the index and one third in high tracking error high orbit satellite portfolios.[2] A representative annualized tracking error for a collection of high tracking error strategies is on the order of 400 basis points. If we apply an expected information ratio (IR) of 0.6, this translates to an average annual excess return of 240 basis points. If we then subtract 72 basis points (30% of the gross excess return) for fees, the asset owner has an expected net return on the collection of high conviction strategies of 168 basis points. This means that an equity portfolio with 65% in the index core (with fees of three basis points), and 35% in the high conviction satellites has an expected after fee excess return of 59 basis points. Rounding this value to 60 basis points, we can think of core and explore as driven by the investor's desire to beat the benchmark by 60 basis points net, per year.

For an investor who needs an average excess return of 60 basis points per year, the core and explore strategy is a relatively simple way to achieve this goal. But what about the role of other, lower orbit, satellite strategies? These lower tracking error strategies are typically quantitatively managed and include enhanced index and factor based (e.g. smart beta) portfolios. While these strategies have lower tracking errors, they also have lower excess returns. Viewed through a mean tracking error lens – minimize expected tracking error subject to the constraint that expected excess return is equal to 60 basis points – these other strategies can only improve the portfolio's overall

[1] See Grinold and Kahn (2000) for modern portfolio theory applied to active management.
[2] Eight public plans in the US representing nearly 300 billion USD in assets have an average asset weighted mix in their domestic equity portfolios of 64% in passive and 36% in active strategies. Frasier-Jenkins et. al (2017) document a large and growing passive allocation for the overall US market.

tracking error characteristics. This obviously improves the information ratio of the portfolio, but the expected excess return is unchanged because it is fixed by the optimization goal of 60 basis points. While an improved information ratio is desirable, it comes at a cost. This cost is not the management fees, as quantitative and factor based strategies tend to have lower fees. Rather, the increased costs are manager search and monitoring costs.[3]

An asset owner focused primarily on achieving an excess return target (e.g. 60 basis points above benchmark) is unlikely to endure the increased searching and monitoring costs if the only improvement is a modest increase in the information ratio of the portfolio. These hidden, or soft costs are higher for active strategies than for rule based factor portfolios, explaining, at least in part, the growing attraction of factor based strategies.

This analysis however, is not the whole story. The analysis suffers from two drawbacks: (i) it assumes that tracking error is an adequate description of active risk, and (ii) there is no role for investment horizon. In this paper we think of active risk as falling short of a target, e.g. the 60 basis points excess return discussed above. This means that the objective is to minimize expected shortfall. And rather than minimizing expected shortfall at a single horizon, we focus on minimizing shortfall over a range of horizons. While we apply this notion of shortfall risk in a relative context to obtain insight into the optimal allocation to a variety of equity strategies, this framework also applies in an absolute return context. Tarlie (2016) contains a simple application of this notion of shortfall risk to the problem of allocating between stocks and bonds.

We find that optimal expected (relative) shortfall portfolios differ substantially from both the 65/35 core and explore portfolio and the portfolios generated using the conventional mean tracking error approach. For the problems we consider in this paper, optimal expected shortfall portfolios lie on the tracking error efficient frontier. Holding the target excess compounding rate constant, as horizon increases, the location of the optimal expected shortfall portfolio moves up and to the right along the frontier. For example, the optimal mean tracking error portfolio with 60 basis points of expected excess return corresponds to an expected shortfall portfolio with a three year horizon.

As horizon increases beyond three years, we show how investors can have their cake and eat it too. Having their cake means that the optimal expected shortfall portfolios have lower expected shortfall than the corresponding optimal mean tracking error portfolios.[4] And, eating it too means that these optimal expected shortfall portfolios also have significantly higher expected surplus and significantly lower shortfall probability than their mean tracking error counterparts. We also show that these results are robust to the specifics of the core and explore example. As long as the target excess compounding rate is less than the excess geometric return of the (relative) growth optimal

[3] Beck et. al. (2016) make this point as well, but they also point out that actively managed strategies can benefit from more flexible trading schedules, thereby reducing trading costs relative to passive funds.
[4] For simplicity we use the term mean variance to refer to mean tracking error throughout the paper.

portfolio, then the reward to risk profile (as defined by surplus/shortfall) of the optimal portfolio improves as horizon grows.

These shortfall, surplus and probability benefits do come at a cost. The cost is a higher tracking error. In this framing, tracking error is not active risk. Rather, it is the *cost* necessary to minimize expected shortfall. Since the portfolios we study in this paper are also mean tracking error efficient, the framework therefore provides a mechanism to identify how much tracking error is necessary to minimize expected shortfall. For example, for a target excess compounding rate of about 60 basis points per year and a portfolio horizon of about 20 years, the tracking error that minimizes expected shortfall is almost twice the tracking error of the corresponding mean tracking error solution. Investors seeking to minimize expected shortfall therefore need to aim significantly higher from both an expected excess return and tracking error perspective than implied by the conventional approach.

Our work has a number of implications for asset owners who care about relative performance. First, asset owners should revisit their equity strategy allocations from the perspective that active risk is expected shortfall over a range of investment horizons. This means paying particular attention to excess return targets and portfolio horizon. Second, asset owners should reevaluate their stance towards tracking error. The tracking error that minimizes expected shortfall depends on portfolio horizon. Asset owners with long horizons can have their cake and eat it too – which means lower expected shortfall but higher expected surplus and lower probability of shortfall – but this requires aiming higher from a tracking error perspective than implied by the conventional mean tracking error approach. In this framing tracking error is the cost necessary to minimize expected shortfall, which we view as active risk. One implication is that using tracking error to proxy for risk, due for example to career risk considerations, is in conflict with a long portfolio horizon, which implies the need to bear the cost of higher tracking error.[5] Third, those asset owners that have adopted the core and explore approach and that have reasonable excess return targets and long portfolio horizons, should allocate away from the passive index to strategies with positive (after fee and transaction costs) expected excess returns. While the soft costs of managing these strategies is clearly higher than managing a passive index, enduring these costs is warranted by the consequences of falling short of their desired excess return targets.

The problems with tracking error as a measure of risk are well known in the literature. Notable papers are Roll (1992) and Jorion (2003), among others. These papers illustrate that mean tracking error optimal portfolios are not optimal with respect to the absolute return and volatility characteristics of the portfolio. Jorion (2003), for example, explicitly advocates for plan sponsors to concentrate on total portfolio volatility instead of tracking error. Portfolio selection based on a relative return perspective maximizes expected excess return subject to a constraint on tracking error and is discussed in detail in Grinold and Kahn (2000). Papers that use tracking error as a

[5] One way to reconcile career risk considerations with expected shortfall is to match the portfolio horizon with the three year timeframe over which managers are typically evaluated.

measure of risk include Blitz and Hottinga (2001), who maximize information ratio for uncorrelated excess returns as a means of determining optimal tracking error allocations, and Markus et. al. (1999), who optimize over various forms of linear tracking error models. Berkelaar et al. (2006) propose a tracking error allocation framework oriented toward institutional investors. In our work, we use tracking error as a means of classifying different strategies, and we focus on expected shortfall as active risk, not on tracking error or portfolio volatility.

Investment horizon plays a central role in this paper. It is well known that optimal portfolios are myopic if: (i) returns are independent and identically distributed, (ii) risk aversion is constant, and (iii) rebalancing is allowed. This is the classical myopic result of Samuelson (1969). However, practitioners typically suggest that younger investors hold more stocks than older investors, advice contrary to the classic myopic result of Samuelson. Thorley (1995) and Hansson and Persson (2000) explain this practitioner intuition by showing that myopia does not hold and stocks become more attractive as horizon increases if portfolios are buy and hold, i.e. rebalancing is not allowed. In this paper we apply an expected shortfall utility function that is linear below the target and flat above the target so that risk aversion is not constant. In this expected shortfall approach, myopia does not hold and higher tracking error strategies become more attractive as horizon increases even if returns are iid and if rebalancing is allowed.

Our work also focuses on the role of factor (e.g. smart beta) strategies. These strategies have recently received a lot of attention. Ang (2014), Chapter 14, provides a good motivation for investors to consider factor investing, and Homescu (2015) provides a comprehensive analysis of factor investing, concentrating on the factors and their combinations. Our paper complements these two studies by providing an allocation framework for how factor portfolios fit with other equity strategies. Our allocation framework differs from that in Carson et. al. (2017), who provide a smart beta glide path within the context of a predefined stock-bond allocation. Their approach is to maximize the Sharpe ratio of the smart beta glide path with a constraint on the tracking error relative to the stock-bond glide path. Keller (2014) builds optimal portfolios using smart beta portfolios based on a variant of maximum Sharpe ratio. Finally, a good review of the actual performance of smart beta ETFs is provided by Glushkov (2016), who also offers a classification of the most popular ETFs.

1. Strategy Allocation

Our basic question is how to allocate between different equity strategies. For concreteness, we focus on the specific goal of outperforming the policy benchmark by 60 basis points per annum (net of fees and transaction costs).

While the details of our core and explore are motivated by US strategy characteristics, our overall conclusions are robust to the specifics. In particular, we will show that as long as the target excess compounding rate is less than the excess geometric return of the (relative) growth optimal portfolio, then the optimal expected shortfall portfolio lies on the tracking error efficient frontier.

The intuition for this result is that as long as the target excess compounding rate is feasible, i.e. there is a portfolio with an expected excess compounding rate that is larger than the target excess compounding rate, then higher tracking error (holding excess return constant) is undesirable and optimal expected shortfall portfolios are mean tracking error efficient. Holding target excess compounding rate constant, as horizon increases the optimal portfolio moves up and to the right along the frontier. But in contrast to the reward to risk ratio as defined by excess return/tracking error, which is either constant or decreasing as the portfolio moves up and to the right along the frontier, the reward to risk characteristics from a surplus-shortfall perspective improve as horizon increases and the portfolio moves along the frontier. It is this improvement in surplus/shortfall reward to risk characteristics as a function of investment horizon that drives our main results.

We begin, in Section 1.1, by articulating our assumptions regarding strategy characteristics, and then in Section 1.2 we apply the conventional mean variance approach to illustrate explicitly that adding these two lower tracking error strategies only modestly improves tracking error and information ratio characteristics. In the presence of hidden and soft costs, these modest improvements in tracking error may not warrant including these two lower tracking error strategies.

In Section 1.3 we consider an alternative to the mean tracking error approach. In this alternative approach we add two new ingredients. First, we define active risk as falling short of outperforming the benchmark by the targeted amount, e.g. 60 basis points per annum. Second, we account for this risk over a range of investment horizons. Specifically, the objective is to minimize expected shortfall relative to the desired excess return over some portfolio horizon.

Our main result, contained in Section 1.4, is that if investment horizon is long enough, then investors can have their cake and eat it too. Having their cake means that expected shortfall is optimal (by construction), and eating it too means that the portfolios also have highly desirable surplus and probability characteristics, even though we only optimize over shortfall.

Furthermore, for the shortfall preferences we consider in this paper, optimal expected shortfall portfolios are also mean tracking error efficient.[6] This means that optimal expected shortfall portfolios lie on the tracking error efficient frontier – they do not represent an improved frontier. Specifying investment horizon and target compounding rate simply defines the location on the frontier that minimizes expected shortfall. Optimizing expected shortfall for a given horizon and target compounding rate therefore determines the appropriate amount of tracking error needed to minimize expected shortfall.

1.1 Strategy Characteristics

[6] In general, optimal expected shortfall portfolios are mean variance efficient as long as target compounding rates are not too high, i.e. lower than the expected geometric return of the growth optimal portfolio, and horizons not too short. See Section 4.2 in Tarlie (2016) for details.

Our focus in this paper is on understanding how to allocate between four stylized equity strategies: pure index, enhanced index, factor (e.g. smart beta), and high conviction. We classify the available strategies by tracking error, from low to high, as summarized in Table 1. The passive index matches the policy benchmark and has no tracking error, the enhanced index has low tracking error, a factor-based strategy has medium tracking error, and high conviction strategy has high tracking error.

Table 1. Classification scheme for investment strategies by tracking error

Tracking Error	Zero	Low	Medium	High
Investment strategy	Passive index	Enhanced Index	Factor Portfolio	High Conviction
Typical tracking error	0%	≲1.5%	1.5% ~ 4%	≳4%

The core and explore strategy combines a large allocation to a passive core with a collection of high conviction strategies. The passive core is simply a market capitalization weighted index and represents the default strategic asset allocation option. The high conviction strategy represents a blend of high tracking error portfolios. These portfolios, termed high conviction because of their demonstrated willingness to bear a high tracking error, tend to be concentrated in a few securities with large deviations from underlying index weights and frequently include meaningful out of benchmark holdings. While an index has no tracking error by definition, the high conviction portfolio represents the other extreme.

The enhanced index strategies are characterized by lower tracking error, and are typically well diversified with many small deviations from benchmark weights, and few, if any, out of benchmark holdings. These portfolios generate modest levels of tracking error by tilting the portfolio, at the level of single stocks, towards a variety of characteristics (a.k.a. factors), such as value, quality or size.

The success of bottom up factor tilting and the ability to replicate some active return streams has naturally led asset owners to consider owning the top down factors directly. The resulting factor portfolios typically have low fees and high levels of transparency. This reduces some of the monitoring and searching costs associated with active strategies. The increased transparency, however, is often at the cost of portfolio efficiency. As a result, although factor and enhanced index strategies have similar modest expected excess returns, factor portfolios tend to have intermediate or medium levels of expected tracking error.

We base our analysis on a set of stylized assumptions to describe the four strategies. These stylized assumptions are motivated by results from the eVestment database that contains returns for a wide array of managers and from a historical estimate of a simple multi factor strategy.[7] While our assumptions are motivated by historical values for the US, our results, which stem from the basic result that reward to risk characteristics from a surplus/shortfall perspective improve as horizon increases, are not sensitive to the specific numerical values that we choose.

Table 2 shows our stylized assumptions for the four assets. We assume that all active strategies charge 30% of the expected gross excess return. This means that the projected fee is 15 basis points for the enhanced index but 72 basis points for the high conviction portfolio. Furthermore, we assume that the smart beta portfolio charges 20% of the expected gross excess return, the lower percentage reflecting the simpler portfolio construction rules.

The gross and net returns in Table 2 are assumed to be net of transaction costs. For the factor strategy, we estimate transaction costs of 0.24%, in line with Chow et. al. (2017) who estimate market impact costs in the neighborhood of 0.20% for multi factor smart beta portfolios.

Table 2. Stylized asset characteristics for passive index, enhanced index, factor portfolio and high conviction portfolio. All returns are annualized geometric. All values are in percentage points, except information ratios and correlations.

	Index (Passive)	Enhanced Index	Factor Portfolio	High Conviction
Gross Excess Return	0	0.50	0.50	2.40
Fees	0.03	0.15	0.10	0.72
Net Excess Return	-0.03	0.35	0.40	1.68
Tracking Error	0	1.25	2.00	4.00
Gross Information Ratio	N/A	0.40	0.25	0.60
Net Information Ratio	N/A	0.28	0.20	0.42
			Excess Return Correlations	
Enhanced Index		1	0.25	0
Factor Portfolio			1	0
High Conviction				1

In terms of correlations, we assume that the high conviction portfolio excess return is uncorrelated with both the factor portfolio and the enhanced index. We assume a correlation of 0.25, however, between the enhanced index and the factor strategies even though the empirical correlation using the eVestment dataset is close to zero. An important difference between the enhanced index strategies and the factor strategies is that the former use a bottom up, stock selection framework, whereas the latter use a top down, factor orientation. Although factor portfolios and enhanced index portfolios may start with similar factors, the application of top down selection of stocks in the smart beta portfolios and the bottom up selection for enhanced index portfolios, as well as

[7] Results in the eVestment database are self reported and suffer from survivorship bias. A similar analysis using the CRSP database on mutual funds finds broadly similar results to the eVestment results. We use self described quantitative managers to guide our assumptions about the Enhanced Index strategy, and self described fundamental managers to guide our assumptions about the High Conviction strategy.

distinct risk management approaches and design decisions, produces different excess return patterns. We nonetheless assume a correlation of 0.25 to reflect the idea that the enhanced index and factor strategies are typically constructed from the same underlying characteristics such as value, quality, and size.

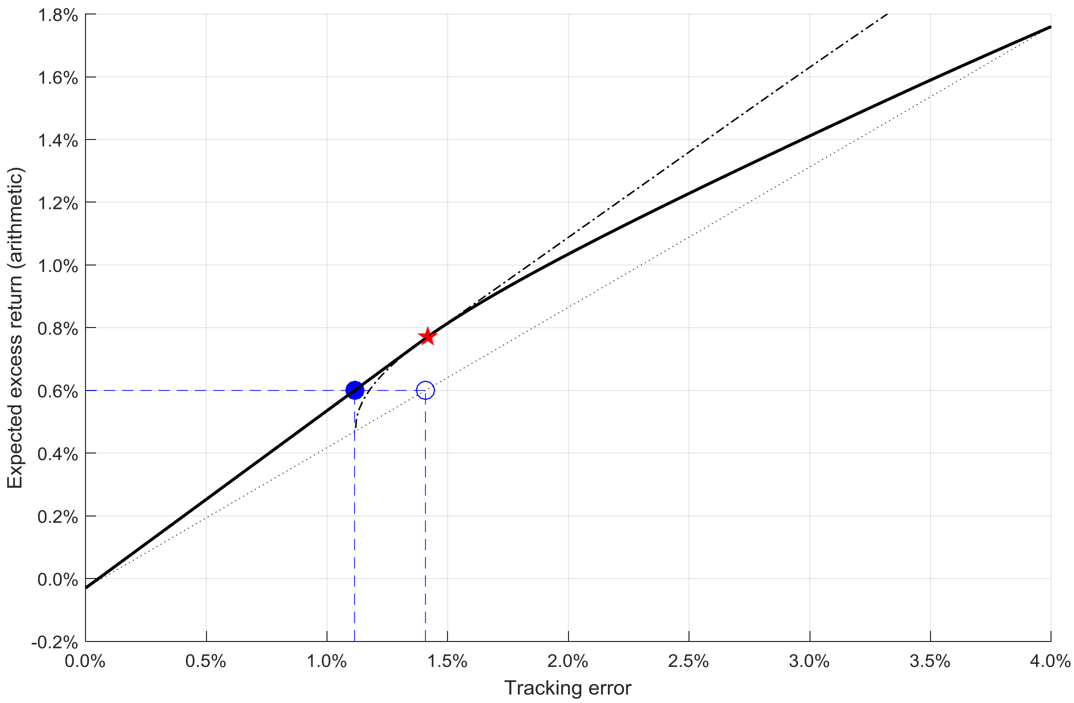
1.2 Mean Tracking Error Optimization

In this section we find the optimal strategy mix based on minimizing expected tracking error subject to the constraint that expected excess return exceeds 60 basis points. We start with the simple two asset problem where assets are the passive index and the high conviction portfolio. Because the passive index is risk free (i.e. zero tracking error) in active space, the tracking error frontier starts just below the origin, reflecting the three basis points of fees, and is linear (the dotted blue line in Figure 1). Every portfolio on the two asset frontier has the same information ratio of 0.43.

In the standard textbook approach, the investor's aversion to tracking error determines the appropriate portfolio on the tracking error efficient frontier. Alternatively, the investor can choose to constrain the optimization to target a desired tracking error or excess return. These three approaches – tracking error aversion, target tracking error, target excess return – are functionally equivalent as investor preferences boil down to a single parameter. Thus, solving for a 60 basis point excess return target produces the prototypical core and explore portfolio with 65% in the passive index, 35% in the high conviction composite, and an annualized tracking error of 141 basis points. This portfolio is illustrated by the open circle in Figure 1 and weights and statistics are shown in the first column of Table 3.

What happens when we add the enhanced index and factor portfolios? Instead of a straight line, the frontier (shown as the solid line) now curves slightly reflecting the increased diversification available. If we solve for the same 60 basis point excess return target, the optimal four asset mean variance portfolio, illustrated by the filled circle, allocates 20% to the index, 23% to the high conviction portfolio, 42% to the enhanced index, and the remaining 15% to the factor portfolio. The four asset optimal mean variance portfolio has the same expected excess return as the core and explore portfolio, but a lower expected tracking error of 112 basis points. This lower tracking error results in a modestly higher net information ratio of 0.54 compared to 0.43. Weights and statistics are shown in the second column of Table 3.

Figure 1. Tracking Error Efficient Frontier. The dotted line is the two asset efficient frontier based on the passive index and the high conviction portfolio. The solid line is the four asset efficient frontier without leverage. The dashed-dotted line is the traditional tracking error frontier without the zero tracking error passive index and allowing for leverage.



A useful way to understand the optimal four asset portfolio is to start with the maximum information ratio portfolio. The maximum information ratio portfolio, denoted by the star in Figure 1, is located at the kink in the efficient frontier that separates the linear and the curved portions of the frontier. As shown in the last column of Table 3, this maximum information ratio portfolio has no weight in the pure index strategy, 52% in the enhanced index strategy, 19% in the factor strategy, and 29% in the high conviction strategy, implying an expected excess return of 78 basis points and tracking error of 142 basis points. All the portfolios with target excess returns below 78 basis points have some combination of the passive index and the maximum information ratio portfolio. Since we do not allow for leverage, if the target expected return is above 78 basis points then the optimal portfolio lies on the portion of the efficient frontier above and to the right of the maximum information ratio portfolio.

Table 3. Portfolio weights and characteristics for the two asset core and explore portfolio, the four asset mean tracking error portfolio (TE), and the maximum information ratio (IR) portfolio. All values are in percentage points, except information ratio.

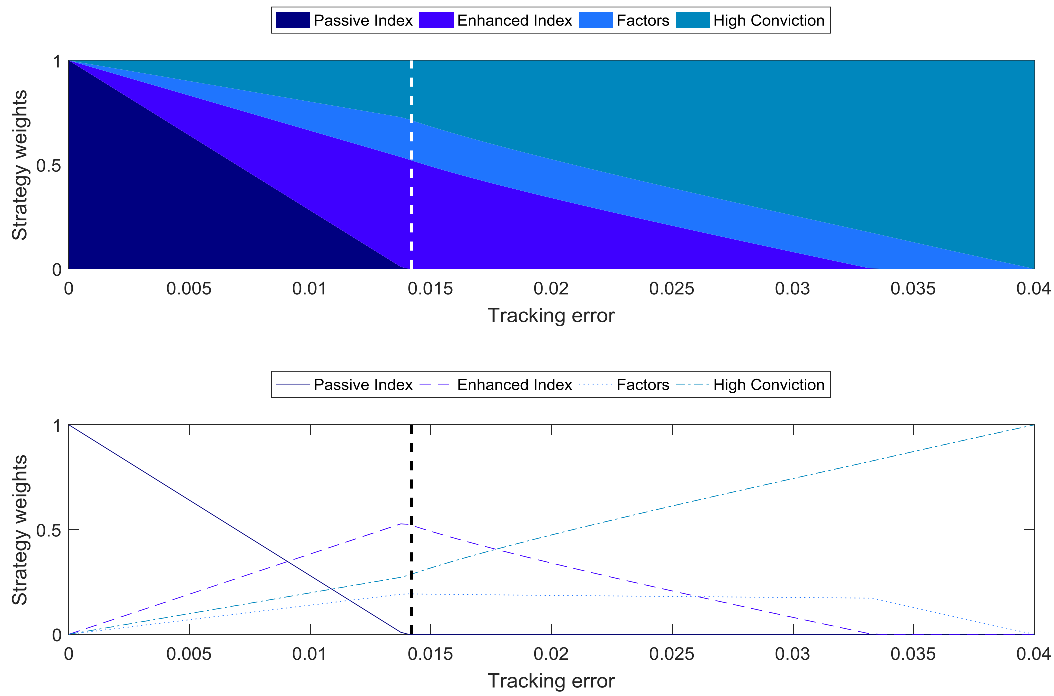
	2 Asset Core and Explore	4 Asset Mean TE	Max IR
Weights			
Index	65	20	0
Enhanced Index	0	42	52
Factor Portfolio	0	15	19
High Conviction Portfolio	35	23	29
Portfolio characteristics			
Net Excess Return	0.60	0.60	0.78
Tracking Error	1.41	1.12	1.42
Net Information Ratio	0.43	0.54	0.54

An interesting feature of the maximum information portfolio is that it is dominated by the enhanced index strategy. The enhanced index strategy is 52% of the maximum information ratio portfolio, compared to 29% for the high conviction strategy. The enhanced index strategy dominates the high conviction strategy even though the net information ratio of the enhanced index strategy is 0.28, compared to 0.40 for the high conviction strategy. Why is this? The intuition is that in the limit of zero correlations, portfolio weight is proportional to mean divided by variance, not mean divided by standard deviation.[8] From Table 2, (net) excess return divided by tracking error squared is 22.4 for the enhanced index strategy, but only 10.5 for the high conviction strategy, explaining the dominance of the enhanced index strategy.

To complete the characterization of the mean tracking error optimal portfolios, Figure 2 shows the optimal strategy weights as a function of tracking error. The top panel shows an area chart, and the bottom panel a line chart. Highlighted in both of these charts is the tracking error of the maximum information ratio portfolio. Examining this figure, we see that as tracking error increases from zero to the tracking error of the maximum information ratio portfolio, the weight in the passive index decreases monotonically from 100% to zero. On the other hand, the weight in the high conviction strategy increases monotonically as tracking error increases. The weights of both the enhanced index and factor portfolios at first also increase with tracking error. But they achieve maximum values at or above the maximum information ratio portfolio, before decreasing to zero as tracking error reaches its maximum. This means that there is a sweet spot in the excess return-tracking error plane where these lower tracking error strategies play an important role. We will see below how this feature of the efficient set of portfolios relates to investment horizon.

[8] While correlations are not zero in our problem, the only nonzero correlation is between the enhanced index and the factor portfolio, and the value is 0.25.

Figure 2: Mean tracking error optimal strategy weights as a function of tracking error. The top chart is an area chart, and the bottom is a line chart that more clearly illustrate how the weights of the four strategies change with tracking error. The dashed vertical line is the tracking error of the maximum information ratio portfolio.



Soft Costs of Active Management

All else equal, lower tracking error, and hence higher information ratio, is desirable. However, all else is rarely equal. For example, the mean tracking error approach does not account for some of the real world costs faced by institutional investors. When an investment manager adds an additional client, the costs of managing a marginal dollar are typically small. By contrast, asset owners face the opposite problem. Each additional active strategy adds to the burden on the investment staff's limited time and attention. The *soft costs* of active management include the time and effort devoted to monitoring managers, as well as the career risk associated with inevitable periods of manager underperformance. Furthermore, industry peer rankings for asset owners tend to emphasize absolute returns instead of risk adjusted returns.[9] Under these conditions, a modestly lower information ratio may well be a price many asset owners are willing to pay to avoid operational complexity.

[9] See NACUBO-Commonfund Study of Endowments 2016 http://www.nacubo.org/Research/NACUBO-Commonfund_Study_of_Endowments/Public_NCSE_Tables.html

1.3 Expected Shortfall as an Objective

If "active risk as tracking error" is the only approach, then we might be satisfied with the conclusion that the benefits of lower tracking error strategies may not be enough to overcome the soft or hidden costs of active management. But active risk as tracking error is not the only approach. Another approach is to think of active risk as falling short of the excess return target. In this case the objective is to minimize expected shortfall relative to this target. And rather than focus on falling short at a single horizon, we allow for shortfall preferences over a range of horizons.

The idea of downside risk is not new and continues to resonate with practitioners and academics alike. There is a rich literature that explores the tails of the return distribution to highlight worst case return scenarios.[10] A common measure of downside risk is expected tail loss. This expected loss is generally defined as the average expected shortfall below some probability threshold. By contrast, expected shortfall – the average expected portfolio value below a desired target *portfolio value* – is the measure of loss that we use. Furthermore, conventional shortfall approaches usually replace excess return variance with shortfall so that the investment objective is mean shortfall rather than mean tracking error. By contrast, because the target excess compounding rate is built into the target relative portfolio value, the objective function that we use naturally incorporates expected excess return, tracking error, target excess return, and investment horizon.

Relative Portfolio Value

The concept of relative portfolio value is central to our development of shortfall as a meaningful objective for an investor trying to outperform a policy benchmark. It is useful to cast the issue in terms of simple balance sheet items by framing the benchmark as the investor's liability (L) and the portfolio as the corresponding asset (A). The standard accounting approach measures equity as the difference between assets and liabilities, i.e. $A - L$. However, for an investor who needs to compound the asset value at a rate greater than the liability, it is more natural to work with the relative portfolio value, i.e. the ratio of assets to liabilities

$$W = \frac{A}{L}. \tag{1}$$

The ratio of assets to liabilities, unlike the difference, more directly captures the effect of compounding over time. To illustrate, suppose for simplicity that the liability is expected to grow at a constant rate l while the asset is expected to grow at a constant rate a. If both the asset and the liability start with the same values at $t = 0$, then the relative portfolio value at time t is given by

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[10] Some notable references include: Roy (1952), which emphasizes the investor's preference for return above a certain level and is closely related to Markowitz (1952); Fishburn (1977) which frames shortfall in a utility theory context; Kahnemann and Tversky (1977), which modifies conventional utility by introducing a reference level and assigning different values to gains and losses; and Bertsimas et. al. (2004) who solve a mean shortfall optimization problem.

$$W = e^{(a-l)*t}\ . \tag{2}$$

For a completely passive portfolio, assets match liabilities perfectly, i.e. $a = l$, and $W = 1$. For active portfolios, relative portfolio value can take on any positive value depending on how the assets have performed relative to the benchmark. For instance, a portfolio with a 2% excess return over a single year has a relative portfolio value of $e^{0.02} \approx\ 1.02$.

In general, the asset and liability growth rates (i.e. geometric returns) *a* and *l* are random variables that depend on time and are not constant. For normally distributed excess returns, the implied target excess compounding (i.e. geometric) rate, in terms of the target excess (arithmetic) return μ^* and portfolio tracking error σ_p, is given by

$$\gamma^* = \mu^* - 0.5 * \sigma_p^2. \tag{3}$$

In our portfolio example, the reference core and explore portfolio has a target excess return of $\mu_p = 60$ basis points per year and annualized tracking error of $\sigma_p = 141$ basis points, implying a target excess compounding rate of $\gamma^* = 59$ basis points per year.

For an investment horizon of one year, we can think of the target excess compounding rate as implying a target relative portfolio value of $W^* = e^{0.0059} \approx 1.0059$. More generally, for horizon T (measured in years) the target relative portfolio value is

$$W^*(T) = e^{\gamma^* T}, \tag{4}$$

which in our example is $e^{0.0059 * T}$. Throughout this paper, we measure time in years.

We assume that excess arithmetic returns for the two asset and four asset optimal mean tracking error portfolios are independent and identically (normally) distributed – iid normal –with a mean of $0.0060 * T$ and tracking errors of $0.0141 * \sqrt{T}$ and $0.0112 * \sqrt{T}$, respectively. Assuming normally distributed excess returns means that relative portfolio value is lognormally distributed.

Expected Shortfall for a Single Horizon

The risk for an investor trying to outperform a policy benchmark is that the relative portfolio value falls below the desired target. Tarlie (2016) provides a framework for specifying how much the investor cares about falling short of their target relative to exceeding it. A useful special case is to measure shortfall risk as the expected percentage deviation of relative portfolio value below the target. As shown in Tarlie (2016) this is analogous to a linear utility function below target and a flat utility function above target.

Accordingly, we measure expected shortfall as the probability weighed sum of the percentage shortfall of the relative portfolio values below the target. If W_T represents relative portfolio value at horizon T, and $W^*(T)$ is the target relative portfolio value (Eq. (4)), then expected shortfall is given by

$$\Phi(T) = \int_0^{W^*(T)} \left(\frac{W^*(T) - W_T}{W^*(T)}\right) P(W_T) dW_T, \tag{5}$$

where $P(\cdot)$ is the lognormal distribution. The probability of shortfall is simply the sum of the probabilities for relative portfolio values below the target. In a related manner, the expected surplus is the probability weighted sum of the percentage surplus of the relative portfolio values above the target, i.e.

$$\Pi(T) = \int_{W^*(T)}^{\infty} \left(\frac{W_T - W^*(T)}{W^*(T)}\right) P(W_T) dW_T, \tag{6}$$

and the probability of surplus is simply the sum of the probabilities for relative portfolio values above the target. While we do not optimize over expected surplus, it is a useful portfolio characteristic that we use in our analysis.

For lognormally distributed relative portfolio values, the integrals for expected shortfall are available in closed form. In particular, for horizon T and target (excess) compounding rate γ^*, the objective function for expected shortfall (see Eq. (13) in Tarlie (2016)) is

$$\Phi(x, \gamma^*, T) = N(z_1) - \left[e^{-z_1 \bar{\sigma}_T + \frac{\bar{\sigma}_T^2}{2}}\right] N(z_2) \tag{7}$$

$$z_1 = \frac{\gamma^* T - \bar{\gamma}_T}{\bar{\sigma}_T}, \quad z_2 = z_1 - \bar{\sigma}_T, \tag{8}$$

where $N(\cdot)$ is the standard cumulative normal. The quantities $\bar{\gamma}_T$ and $\bar{\sigma}_T^2$ in Eqs. (7) and (8) are the mean and variance, respectively, of the log of relative portfolio value at horizon T, i.e. $\bar{\gamma}_T = E[\ln W_T]$ and $\bar{\sigma}_T^2 = Var(\ln W_T)$. We see from Eq. (7) that the objective function Φ naturally incorporates notions of mean, variance, target return, and investment horizon.

Expected shortfall has two terms that account for both the probability and magnitude of shortfall. The first term, $N(z_1)$, is the probability of shortfall. So minimizing expected shortfall means minimizing the probability of shortfall, in part. Since the cumulative normal is increasing in z_1, minimizing the probability of shortfall means minimizing z_1; this is essentially Roy's Safety First Criterion (Roy (1952)), which is closely related to Markowitz's mean variance objective. The second term, which can be written $E[W_T]N(z_2)$, accounts for the magnitude of shortfall. This term comes with a minus sign, so minimizing expected shortfall is achieved, in part, by increasing the value of $E[W_T]N(z_2)$. The term $E[W_T] = e^{\bar{\gamma}_T + \bar{\sigma}_T^2/2}$ favors higher expected excess return and higher tracking error, but this term is discounted by $N(z_2)$, which is a decreasing function of both expected excess return and tracking error. Notice also that if $\gamma^* < \bar{\gamma}_T/T$, then as the horizon T goes to infinity z_1 goes to minus infinity. This means that if the target compounding rate is achievable, then the probability of shortfall goes to zero in the very long run. Thorley (1995), page 74, shows a similar result when comparing the long term value of a risky fund and a risk free fund.

The net effect is that expected shortfall is always decreasing in total expected excess return $\gamma^* T - \bar{\gamma}_T$. In this paper we only consider preferences such that decreasing total tracking error is always desirable, i.e. $\partial\Phi/\partial\bar{\sigma}_T > 0$.[11] This condition means that all optimal expected shortfall portfolios are also mean tracking error efficient in that they lie on the tracking error efficient frontier. We will see below that setting the target excess compounding rate γ^* and the investment horizon T amounts to identifying the location on the efficient frontier that minimizes expected shortfall. This expected shortfall framework therefore provides an alternative to the traditional methods of choosing the appropriate tracking error efficient portfolio based on either tracking error aversion, or target expected excess return, or target tracking error.

The expected shortfall objective function in Eq. (7) warrants two additional points. First, expected shortfall results from a standard application of expected utility theory and the explicit evaluation of a standard integral.[12] Second, the functional form of Eq. (7) is familiar. It resembles the formula for the price of a European option. However, the resemblance, which arises because the utility of shortfall is assumed linear in (relative) portfolio value below the target, and flat above the target just like the payoff of a put option, is only superficial.[13]

The two quantities $\bar{\gamma}_T$ and $\bar{\sigma}_T^2$ (see Eqs. (7) and (8)) depend on the model of asset returns and on the term structure of portfolios, i.e. the portfolios over the entire investment horizon. We assume that expected (log) excess returns $\bar{\alpha}_i$ and tracking error $\sigma_{Ri} (> 0)$ are constant so that the dynamics of log excess returns for strategy i follow

$$d\ln R_i(t) = \bar{\alpha}_i dt + \sigma_{Ri} dB_i(t), \tag{9}$$

[11] Expected shortfall is monotonically increasing in total tracking error $\bar{\sigma}_T$, but only if the probability of exceeding the target is greater than 50%, i.e. $z_1 < 0$. The condition $z_1 < 0$ corresponds to the existence of a portfolio with expected excess compounding rate that is larger than the target compounding rate. If the probability of exceeding the target is less than 50%, then for values of total tracking error below a critical value it is possible for expected shortfall to be decreasing in total tracking error, i.e. increased total tracking is desirable; see Section 4.2 of Tarlie (2016) for more details.

[12] For expected shortfall, the integral is given in Eq. (5). The objective function in Eq. (7) results from evaluating the integral in Eq. (5) assuming that wealth is lognormally distributed.

[13] The economic interpretation of the formula for the expected shortfall objective is very different from that of the option pricing formula. The option pricing formula embeds fundamental notions of no arbitrage and replicating portfolios. By contrast, the expected shortfall objective embeds relative portfolio value and the associated attitudes to shortfall and surplus.

where $dB_i(t)$ are temporally uncorrelated Brownian increments. Cross sectional excess return correlations ρ_{ij} are defined by $E\big[dB_i(t)dB_j(t)\big] = \rho_{ij}dt$. If $x(t)$ is the time series vector of portfolio weights for $t \in [0, T]$, where $t = 0$ is the "here and now" and $t = T$ is the investment horizon, then we have

$$\bar{\sigma}_T^2 \; = \int_0^T dt \sum_{ij} x_i(t) x_j(t)\, \sigma_{Ri}\sigma_{Rj}\rho_{ij} \tag{10}$$

$$\bar{\gamma}_T = \int_0^T dt \sum_i x_i(t) \left(\bar{\alpha}_i + \frac{\sigma_{Ri}^2}{2}\right) - \frac{\bar{\sigma}_T^2}{2}, \tag{11}$$

where $x_i(t)$ is the weight in asset i at time t. The portfolio weights are normalized to one so that $\sum_i x_i(t) = 1$ for $t \in [0, T]$.

Expected Shortfall for Multiple Horizons

The expected shortfall objective function in the previous section is for a single investment horizon. In general, however, an investor who is averse to shortfall risk cares about this risk over a range of horizons. We follow the standard approach of time discounting so that the multi horizon objective function has the form

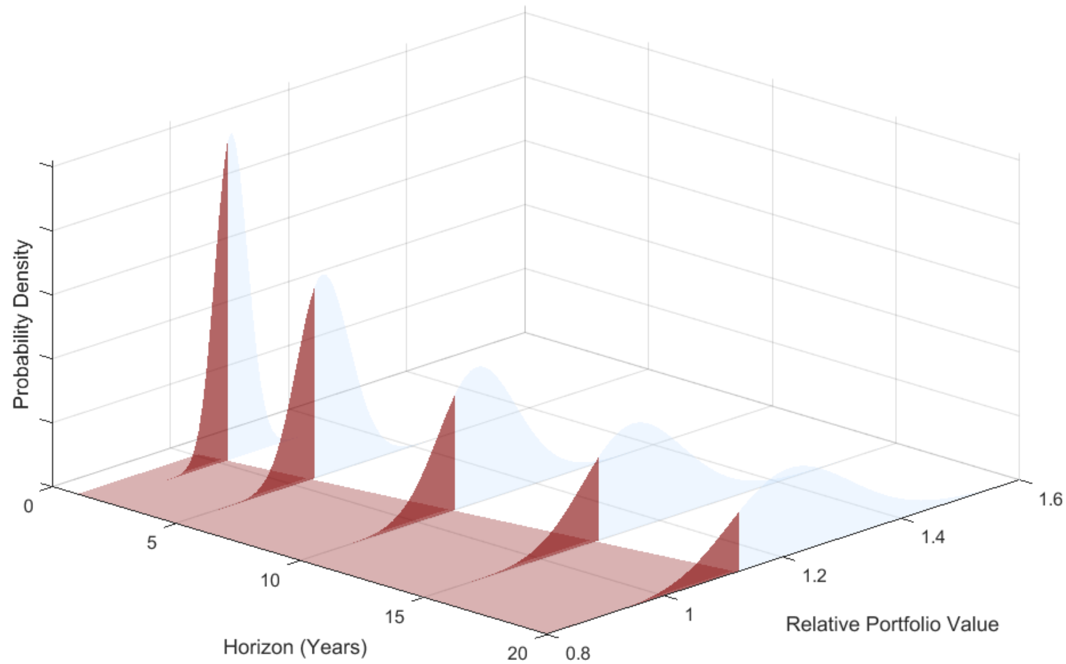
$$\overline{\Phi}(x, \gamma^*, T_1, T_2) \; = \; \int_{T_1}^{T_2} dT \;\, e^{-\beta T}\, \Phi(x, \gamma^*, T), \tag{12}$$

where β is the time preference parameter. The quantity $x(t)$ is the time series vector of portfolio weights, i.e. the term structure of portfolios, for $t \in [0, T_2]$.

In this formulation of the multi horizon objective, the investor cares about shortfall for all horizons between times T_1 and T_2. We can think of times less than T_1 as the patient phase – the investor does not care about shortfall up to time T_1 – and times between T_1 and T_2 as the target phase – the investor cares about falling short of the target.

The plot in Figure 3 illustrates the basic idea for $T_1 = 1$ and $T_2 = 20$. The red shading in the time-wealth plane indicates that for these times the investor cares about falling short of the relative portfolio value, with the upper boundary of this region defined by $W^*(T) = e^{\gamma^* T}$.

Figure 3: Relative portfolio value probability density for different horizons, $\boldsymbol{T} = \mathbf{2}, \mathbf{5}, \mathbf{10}, \mathbf{15}$, and $\mathbf{20}$ years. Dark shaded areas are the relative portfolio values below the desired target $\boldsymbol{W}^*(\boldsymbol{T}) = \boldsymbol{e}^{\mathbf{0.0059} * \boldsymbol{T}}$ for $\boldsymbol{T_1} = \mathbf{1}$ and $\boldsymbol{T_2} = \mathbf{20}$. Probability densities are illustrated assuming a geometric portfolio excess return of 100 basis points and 192 basis points of tracking error.



1.4 Optimal Expected Shortfall Portfolios

We now have all the elements necessary to build optimal portfolios that minimize expected shortfall for different target compounding rate and horizon preferences. We will find that for a given target excess compounding rate, the reward to risk characteristics from a surplus/shortfall perspective increase as horizon increases.

The multi horizon objective function given in Eq. (12) depends on $x(t)$, the term structure of portfolios for times between $t = 0$ and $t = T_2$. Optimizing Eq. (12) generates the optimal term structure of portfolios. But for our purposes the portfolio that matters is $x(0)$, the "here and now" portfolio. The portfolios for $t > 0$ are optimal, conditional on information known at $t = 0$. Consequently, these portfolios reflect the uncertainty of the evolution of the relative portfolio value. As time passes and relative portfolio values become known, the investor re-optimizes and rebalances to reflect this new information; see the Appendix and Section 4.3 in Tarlie (2016) for additional discussion.

Fixed Target Excess Compounding Rate

We start by considering a long portfolio horizon $T_2 = 20$ and a short patient phase of one year so that $T_1 = 1$ for fixed target excess compounding rate $\gamma^* = 0.59\%$. This corresponds to a long portfolio horizon where the investor cares about shortfall over almost the entire portfolio horizon. We also set the time preference parameter $\beta = 0$ to express the preference that shortfall at long horizons matters just as much as shortfall over short horizons. The results are robust to this choice as positive values of β reduce the impact of shortfall at longer horizons thereby effectively reducing the investment horizon.

Table 4 shows the optimal expected shortfall portfolio $x^*(0)$ and associated portfolio statistics for a target compounding rate of 59 basis points. The left panel shows the weights for the four asset optimal expected shortfall portfolio, the four asset optimal mean variance portfolio, and the two asset core and explore portfolio. The main difference between the four asset optimal expected shortfall portfolio and the four asset optimal mean variance portfolio is that the optimal mean tracking error portfolio has a 20% weight in the index, whereas the optimal expected shortfall portfolio has zero weight in the index. Most of this 20% weight goes to the high conviction strategy, which absorbs 15 of the 20 percentage points; the remaining five going to the factor and enhanced index strategies.

Table 4. Comparison of weights and characteristics for the two asset core and explore (CE), four asset mean tracking error (MTE), and four asset expected shortfall (ESF) portfolios. The target excess compounding rate is 59 basis points, and the horizons are $\boldsymbol{T_1 = 1}$ and $\boldsymbol{T_2 = 20}$. All values are in percentage points except for IR, which is the ratio of expected alpha to tracking error. Expected alpha and TE are in annualized units. All return statistics are annualized geometric returns.

	Weights				Statistics					
	Index	Enhanced Index	Factors	High conviction	Expected alpha	TE	IR	ESF	ESP	PSF
4 asset ESF	0	44	19	37	1.00	1.92	0.52	0.85	5.53	33
4 asset MTE	20	42	16	22	0.59	1.12	0.53	1.34	1.41	50
2 asset CE	65	0	0	35	0.59	1.41	0.42	1.68	1.79	50

Legend: TE = tracking error, IR = information ratio, ESF = expected shortfall, ESP = expected surplus, PSF = probability of shortfall.

The right hand panel in Table 4 shows both shortfall/surplus and traditional statistics for these three portfolios. The four asset optimal expected shortfall (ESF) portfolio has an average expected shortfall over the entire investment horizon of 0.85%. This compares to 1.34% for the four asset optimal mean tracking error portfolio and 1.68% for the two asset optimal mean tracking error portfolio. This result is not a surprise, after all, the optimal expected shortfall portfolio has the lowest expected shortfall by optimization.

The striking feature of the results in Table 4 is that the average expected surplus (ESP) is substantially higher and the average probability of shortfall (PSF) is substantially lower for the expected shortfall optimal portfolio, even though it is only expected shortfall that is optimized. In particular, the average expected surplus for the four asset ESF portfolio is 5.53%, substantially above the 1.41% and 1.79% values for both the two asset core and explore and the four asset optimal mean tracking error portfolios. This means that the reward to risk characteristics – from a horizon sensitive surplus/shortfall perspective rather than an expected single period expected excess return/tracking error perspective – are dramatically improved. By contrast, the traditional measure of reward to risk, the information ratio, is essentially the same for the four optimal expected shortfall and four asset mean tracking error portfolios. Furthermore, the average probability of shortfall for the four asset ESF portfolio is 33%, substantially below the 50% average probability for the optimal mean tracking error portfolios.

The improved shortfall, surplus, and probability statistics of the four asset optimal ESF portfolio relative to the four asset optimal mean tracking error portfolio comes at a cost. The cost is increased tracking error. The expected tracking error for the four asset optimal ESF portfolio is 1.92%, compared to 1.12% for the four asset optimal mean tracking error portfolio.

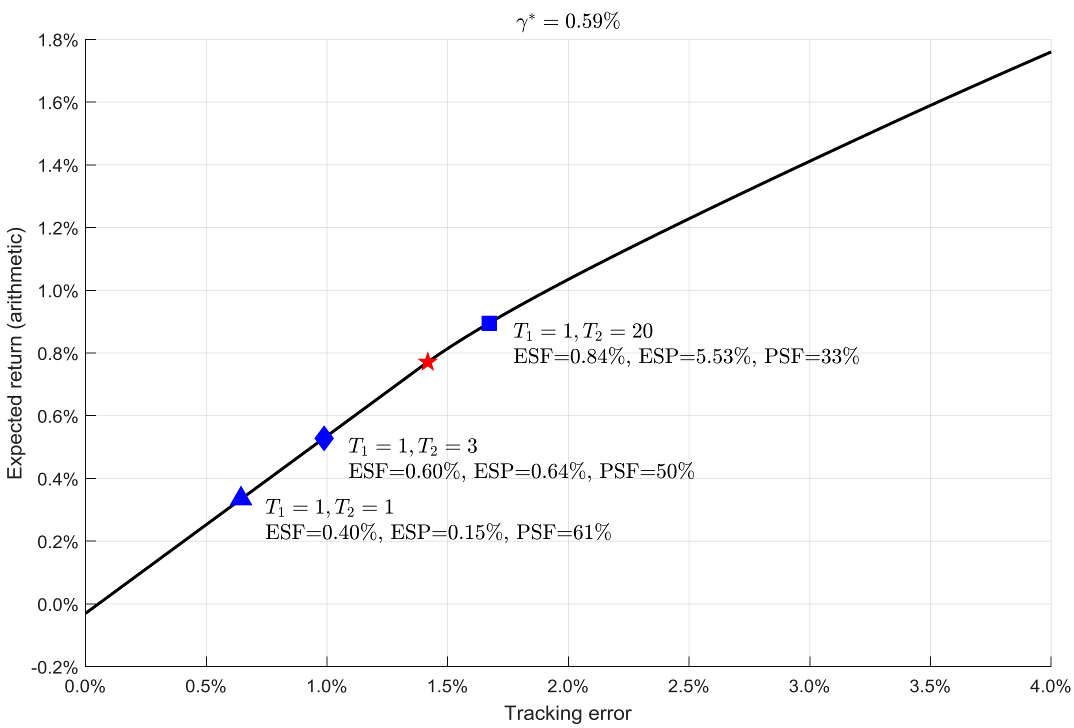
Having Your Cake and Eating It Too (if your horizon is long enough)

Our main result is that investors willing or able to extend their investment horizon can have their cake – low expected shortfall – and eat it too – high expected surplus and low probability of shortfall. Of course, these desirable attributes come at the cost of higher tracking error.

To illustrate this result, let us hold the target excess compounding rate γ^* constant at 59 basis points and fix the patient phase at $T_1 = 1$. As discussed above in the text following Eq. (7), for the problems we consider in this paper optimal expected shortfall portfolios are mean tracking error efficient. This means that holding the target compounding rate constant and increasing the portfolio horizon T_2 is equivalent to moving along the tracking error efficient frontier from lower left to upper right, i.e. as horizon increases both the expected return and the tracking error increases. Note that the four asset optimal mean tracking error portfolio with expected excess return of 0.60% corresponds, approximately, to a three year portfolio of $T_2 = 3$ years.

Figure 4 illustrates this idea by plotting the single period efficient frontier along with the location of the optimal here and now expected shortfall portfolios for $T_2 = 1, 3,$ and 20 years. The average shortfall and surplus characteristics are indicated for each of the three points on the chart. Inspecting these characteristics, we see that as horizon increases the risk return characteristics from a shortfall/surplus perspective improve dramatically. For example, for $T_2 = 1$ the ratio of the average expected surplus to average expected shortfall is 0.15/0.40=0.38 and the average probability of shortfall is 61%. Increasing the horizon to $T_2 = 3$ this ratio of reward to risk improves to 0.64/0.60=1.06 and the average probability of shortfall falls to 50%. Increasing the horizon further to $T_2 = 20$, the horizon for the results in Table 4, the reward to risk ratio improves dramatically to 5.53/0.84=6.6 and the average shortfall probability falls to 33%.

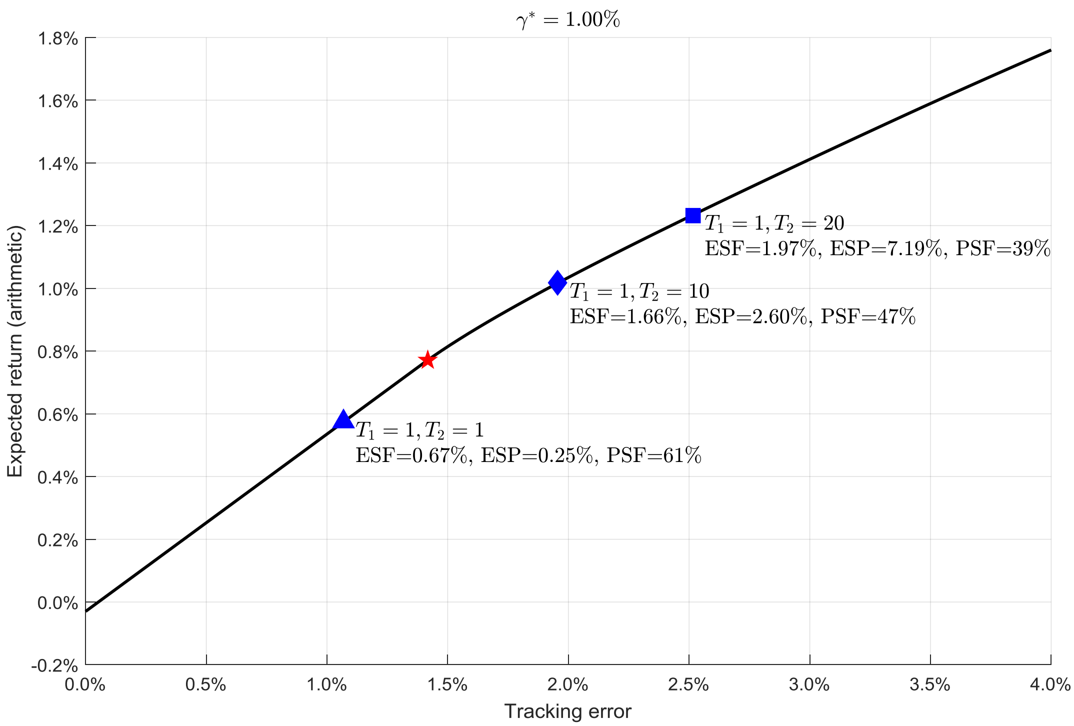
Figure 4: Four asset tracking error efficient frontier (solid line). The triangle, diamond, and square symbols correspond to optimal expected shortfall (ESF) portfolios with fixed target excess compounding rate $\gamma^* = 0.59\%$, $T_1 = 1$ year and horizon preferences $T_2 = 1, 3, 20$ years.



The improvement in the reward to risk ratio from the surplus/shortfall perspective as horizon increases and the location of the optimal here and now portfolio moves up and to the right along the tracking error efficient frontier is different from how the reward to risk profile changes from the excess return/tracking error perspective. As portfolios move up and to the right along the tracking error efficient frontier, the information ratio, the conventional measure of (excess) reward to risk, is constant for points below the maximum information ratio portfolio, and then decreases as the location moves further up and to the right. This behavior is opposite to how the surplus/shortfall ratio changes as portfolios move along the efficient frontier.

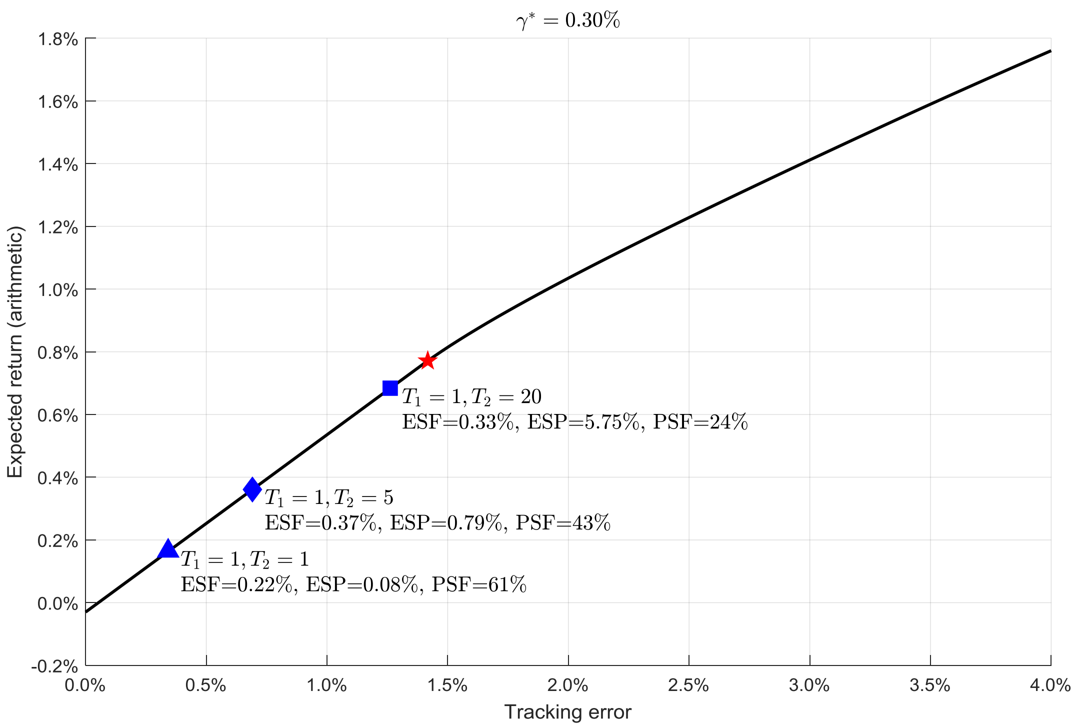
Figures 5 and 6 illustrate how the optimal ESF portfolios move along the efficient frontier for fixed target excess compounding rates of 0.30% and 1.00%. Examining these figures, we see that this improvement in reward to risk from the surplus/shortfall perspective is robust to the choice of target excess compounding rate. This robustness holds as long as the target excess compounding rate is below the excess geometric return of the excess growth optimal portfolio. Furthermore, for both target excess compounding rates of 0.30% and 1.00% we see the same pattern as for the target excess compounding rate of 0.59% – as portfolio horizon T_2 increases, the optimal location on the efficient frontier moves up and to the right. We also see that the risk return characteristics, from an expected shortfall/surplus perspective, improve in that the average expected surplus grows at a faster rate than average expected shortfall, and the average probability of shortfall declines.

Figure 5: Four asset tracking error efficient frontier (solid line). The triangle, diamond, and square symbols correspond to optimal expected shortfall (ESF) portfolios with fixed target excess compounding rate $\gamma^* = 1.0\%$, $T_1 = 1$ year and horizon preferences $T_2 = 1, 10, 20$ years.



Finally, we can use Figure 2, which shows strategy weights as a function of portfolio tracking error, to understand how the strategy weights vary with target excess compounding rates and horizon. In particular, if the combination of target excess compounding rate and horizon is such that the optimal expected shortfall is located near the maximum information ratio portfolio, then the enhanced index fund will have the most weight. As horizon increases, however, the weight in the enhanced index fund declines, being replaced by the high conviction strategy, while the weight in the factor strategy stays roughly constant, except for tracking errors approaching the tracking error of the high conviction strategy. The intuition for this non monotonic behavior is that for zero portfolio tracking error the passive index dominates, whereas for large tracking error the high conviction strategy dominates. But since the low and medium tracking error strategies play an important role in the maximum information ratio portfolio, which has moderate tracking error, their weights as a function of portfolio tracking error must be non monotonic.

Figure 6: Four asset tracking error efficient frontier (solid line). The triangle, diamond, and square symbols correspond to optimal expected shortfall (ESF) portfolios with fixed target excess compounding rate $\gamma^* = 0.30\%$, $T_1 = 1$ year and horizon preferences $T_2 = 1, 5, 20$ years.



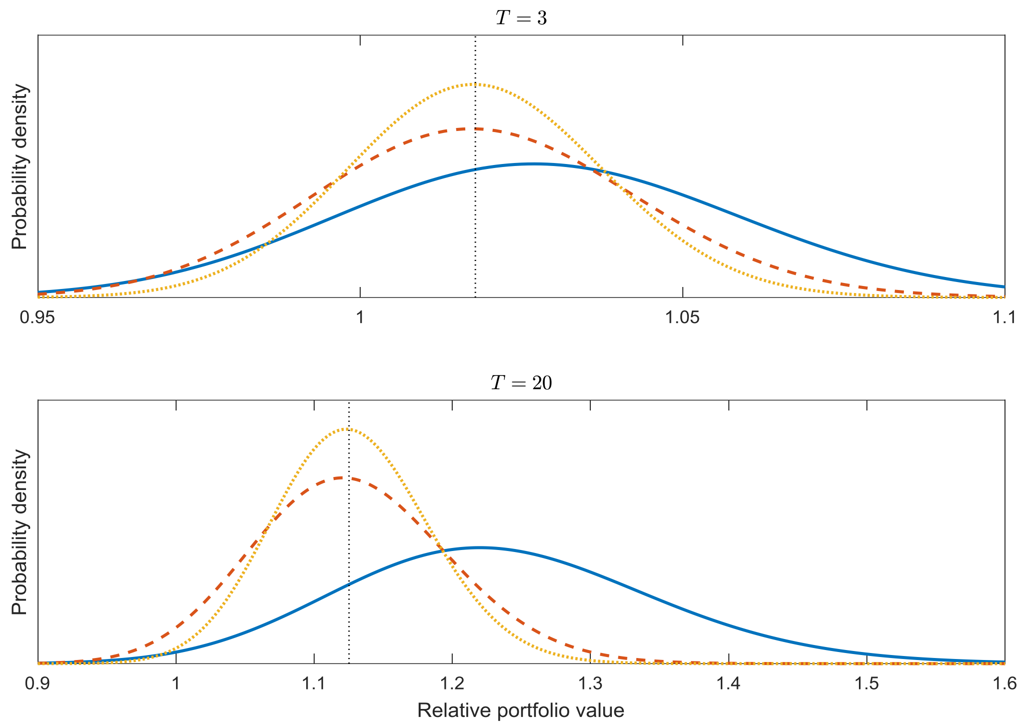
A Comparison of the Distributions of Relative Portfolio Values

The optimal expected shortfall portfolios minimize the expected shortfall objective function given by Eq. (12). While the optimal expected shortfall portfolio minimizes the truncated first moment of the relative portfolio value distribution, it is useful to compare the full distributions for the core and explore, four asset mean tracking error, and four asset expected shortfall portfolios.

The optimal expected shortfall portfolio has left tails that compare favorably with the two asset core and explore portfolio for shorter horizons and with the four asset mean tracking error portfolio for longer horizons. To visualize the full range of outcomes for the two asset core and explore, four asset optimal mean tracking error, and four asset optimal expected shortfall portfolios, Figure 7 plots the distributions of relative portfolio values for horizons of three (top panel) and 20 years (bottom panel). The horizons of three and 20 years correspond to the horizons in the axis coming out of the page in Figure 3, and for the optimal expected shortfall portfolio the target excess compounding rate is 0.59%, $T_1 = 1$, and $T_2 = 20$ years.

In the top panel of Figure 7, which shows the distributions of relative portfolio values at the three year horizon, we see that even though the optimal expected shortfall portfolio has a higher tracking error, the left tail of the distribution effectively matches the left tail of the distribution for the core and explore portfolio. The reason is that the distribution of the optimal expected shortfall portfolio has shifted enough to the right to counteract the increased dispersion.

Figure 7. Probability density of relative portfolio values for the two asset core and explore (dashed line), four asset mean tracking error (dotted line) and four asset ESF (solid line) for horizons of three and 20 years and target compounding rate of $\mathbf{0.59}\%$. Also shown (dotted vertical line) is the relative portfolio value target $\boldsymbol{W}^*(\boldsymbol{T}) = \boldsymbol{e}^{\mathbf{0.0059}*\boldsymbol{T}}$.



In the bottom panel of Figure 7 we see the same basic picture as for the three year horizon, except that now the left tail of the distribution for the optimal expected shortfall portfolio matches the left tail of the distribution of the four asset optimal mean tracking error portfolio. And the left tail of both of these distributions are substantially to the right of the left tail of the core and explore portfolio. The optimal expected shortfall portfolio therefore has a short horizon distribution with a left tail that compares favorably with the two asset core and explore portfolio, and a long horizon distribution with a left tail that compares favorably with the four asset mean variance portfolio.

Conclusion

We study the basic problem of allocating amongst a set of equity strategies given a policy benchmark. The fact that many institutional plans today allocate about two thirds of the portfolio to the benchmark and the remainder to a collection of higher tracking error strategies (the core and explore model), reveals an investor preference for a modest target excess return, e.g. ~0.60% per year. Although lower tracking error strategies, such as quantitatively driven enhanced index and factor portfolios, can lower the tracking error and improve the information ratio, these benefits may not outweigh the increase in soft costs, such as finding and monitoring costs, not to mention additional career risk.

However, the tracking error and information ratio analysis has two main shortcomings. First, tracking error is not the only measure of active risk. Second, there is no accounting for investment horizon. We address these two issues by defining active risk as expected shortfall, in a relative sense. Using this definition, we find that asset owners with reasonable target excess returns and long investment horizons can benefit in terms of both expected shortfall and surplus by incorporating a combination of low, medium, and high tracking error strategies. This means that asset owners should be willing to bear the soft costs associated with these lower orbit strategies because the consequences of falling short are far more meaningful than suggested by the modest tracking error and information ratio benefits.

Appendix

To optimize expected shortfall we discretize time into N_{T_2} discrete units of size Δt. The portfolio weight for asset i at time $t_j = j\Delta t$ for $j \in [0, \dots, N_{T_2} - 1]$ is denoted x_{ij}. Imposing the constraint that portfolio weights sum to one at each time means that $\sum_i x_{ij} = 1$. Discretizing the integral in Eq. (12) leads to an objective function that depends on the set of portfolio weights $\{x_{ij}\}_{j=0}^{N_{T_2}-1}$ over the entire investment horizon, i.e.

$$\overline{\Phi}\left(\{x_{ij}\}_{j=0}^{N_{T_2}-1}\right) \sim \sum_{k=T_1/\Delta t}^{T_2/\Delta t} e^{-\beta T_k}\, \Phi\left(\{x_{ij}\}_{j=0}^{N_{T_k}-1}, \gamma^*, T_k\right),$$

where $T_k = k\Delta t$. Accounting for the constraint that at each time portfolio weights sum to one, the multiperiod objective function for a selection problem with four strategies depends on $3N_{T_2}$ decision variables. In general, the objective function is highly nonlinear and analytic solutions are not possible, except in simplified circumstances (see Section 5.1 in Tarlie (2016) for an example). Thus, we optimize numerically using a generic, off the shelf solver.[14]

The optimization process generates a term structure of optimal portfolios, $x_{ij|0}^*$ for $j \in [0, \dots, N_{T_2} - 1]$. The "|0" in the subscript indicates that this optimal solution is conditional on information known at t_0. The portfolio $x_{i0|0}^*$ is the "here and now" portfolio. This is the portfolio that is relevant to the investor at t_0 because it is the portfolio they will own. While the portfolios $x_{ij|0}^*$ for $j \geq 1$are optimal, they are only optimal conditional on time t_0 information – they are not optimal conditional on time t_j information for $j \geq 1$. Therefore, at each time t_ℓ the investor should re-optimize to generate the term structure of optimal portfolios $x_{ij|\ell}^*$ for $j \geq \ell$ and invest in the portfolio $x_{i\ell|\ell}^*$.

[14] We use fmincon in the Matlab Optimization Toolbox. This method is suitable for constrained, nonlinear problems.

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